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A generalization of Gruenhage's example

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ABSTRACT

We construct, assuming the continuum hypothesis, an example of nonmetrizable n-dimensional Cantor manifold X_n $(n \in N)$ with the following properties: 1) X_n^k is hereditarily separable for all $k \in N$; 2) $X_n^k \setminus \Delta_k$ is perfectly normal for every $k \in N$; 3) the space $\mathcal{F}(X_n)$ is hereditarily normal for every seminormal functor $\mathcal{F}$ that preserves weights and one-to-one points and such that $sp(\mathcal{F}) = \{1, k\}$; in particular, X_n^2 and $\lambda_3 X_n$ are hereditarily normal. This example is a generalization of famous Gruenhage's example given in Gruenhage and Nyikos (1993) [4].

In 1993, assuming the continuum hypothesis *CH*, G. Gruenhage constructed [4] an example of nonmetrizable compact space Y for which Y^2 is hereditarily separable and hereditarily normal, and $Y^2 \setminus \Delta$ is perfectly normal. In 2008 A.V. Ivanov and E.V. Kashuba [6] constructed under *CH* an example of nonmetrizable compact space X with the following properties: 1) X^k is hereditarily separable for all $k \in N$; 2) $X^k \setminus \Delta_k$ is perfectly normal for every $k \in N$, where Δ_k is the generalized diagonal of X^k, i.e., the set of points with at least two equal coordinates; 3) the space $\mathcal{F}(X)$ is hereditarily normal for every seminormal functor $\mathcal{F}$ that preserves weights and one-to-one points and such that $sp(\mathcal{F}) = \{1, k\}$, in particular, X^2 and $\lambda_3 X$ are hereditarily normal where λ the superextension functor. Gruenhage's technique gives a zero-dimensional example Y. The space X constructed in [6] is also zero-dimensional. The construction of X is based on Fedorchuk's method of resolutions and inverse systems (see [1,2]). In present paper we modify this method to construct under *CH* the n-dimensional Cantor manifold X_n with properties 1)–3) above for every $n \in N$. The main idea of this modification is taken from [5] (see also [2]).

A covariant functor $\mathcal{F}$ in the category of compact spaces and continuous mappings is called *seminormal* if $\mathcal{F}$ preserves the singleton and empty set, monomorphic, continuous and preserves intersections (see [3,6]). All functors in this paper we assume to be seminormal. Let $\mathcal{F}$ be a functor and $a \in \mathcal{F}(X)$; the *support supp*(a) is defined as

$$supp(a) = \bigcap \{Y \subset X\colon a \in \mathcal{F}(Y)\}.$$

Given $k \in N$, put

$$\mathcal{F}_k(X) = \{a \in \mathcal{F}(X)\colon |supp(a)| \leqslant k\}.$$

For every $k \in N$, $\mathcal{F}_k$ is a seminormal subfunctor of $\mathcal{F}$. Since $\mathcal{F}_1(X) = X$, we may assume that X is a subspace of $\mathcal{F}(X)$ (see [3]). Put

$$\mathcal{F}_{kk}(X) = \{a \in \mathcal{F}(X)\colon |supp(a)| = k\}.$$

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The *degree spectrum* of $\mathcal{F}$ is the set

$$sp(\mathcal{F}) = \{k\colon k \in N,\ \mathcal{F}_{kk}(k) \neq \emptyset\}.$$

The functor $\mathcal{F}$ *preserves one-to-one points* if given some mapping $f : X \to Y$ and $y \in Y$ with $|f^{-1}(y)| = 1$ the mapping $\mathcal{F}(f)$ is also one-to-one at point y, i.e. $|(\mathcal{F}(f))^{-1}(y)| = 1$ (see [6]). The functor $\mathcal{F}$ *preserves weights* if $w(\mathcal{F}(X)) = w(X)$ for every infinite X.

A compact space X is said to be an n-dimensional Cantor manifold if $\dim X = n$ and each partition P in X has $\dim P \geqslant n - 1$.

All spaces are assumed to be Tychonov. For a mapping $f : X \to Y$ and a set $A \subset X$ we put

$$f^{\sharp}(A) = Y \setminus f(X \setminus A).$$

A mapping $f : X \to Y$ is called fully closed at the point $y \in f(X)$ (see [1]) if for any finite cover $\{U_1, \dots, U_m\}$ of $f^{-1}(y)$ by open sets in X the set

$$\{y\} \cup \bigcup_{i=1}^{m} f^{\sharp}(U_i)$$

is a neighborhood of y. A mapping f is called *fully closed* if f is fully closed at every point $y \in Y$.

Theorem *(CH). For every $n \in N$ there is a nonmetrizable compact space X_n such that*

(1) *X_n is n-dimensional Cantor manifold*;
(2) *X_n^k is hereditarily separable for all $k \in N$*;
(3) *if F is a closed subset of X_n^k and $Cl(F \setminus \Delta_k) = F$ then F is a G_δ-set in X_n^k ($k \in N$)*;
(4) *$X_n^k \setminus \Delta_k$ is perfectly normal for every $k \in N$*;
(5) *the space $\mathcal{F}_k(X_n)$ is hereditarily separable and $\mathcal{F}_{kk}(X_n)$ is perfectly normal for every seminormal functor $\mathcal{F}$ that preserves weights and each $k \in sp(\mathcal{F})$*;
(6) *the space $\mathcal{F}(X_n)$ is hereditarily normal for every seminormal functor $\mathcal{F}$ that preserves weights and one-to-one points and such that $sp(\mathcal{F}) = \{1, k\}$; in particular, X_n^2 and $\lambda_3 X_n$ are hereditarily normal.*

Proof. Let A be a set and $E \subset A^k$, $k \in N$. We say that E is *in general position* in A^k if each projection $p_i : A^k \to A$ of a product A^k onto i-factor ($i = 1, \dots, k$) is one-to-one on E and $p_i E \cap p_j E = \emptyset$ for $i \neq j$.

Lemma. *Let $f : X \to Y$ be a mapping of a set X onto metrizable continuum Y such that only one point $a \in Y$ has a nontrivial inverse image and $|f^{-1}(a)| = c$. Let $\{E_k\}$ be a countable family of sets such that E_k for every k is in general position in some Y^{n_k}, where $n_k \in N$, and coordinates of all points of E_k are distinct from a. Then we can topologize X so that*

1) *X is a metrizable continuum*;
2) *$f^{-1}(a)$ is homeomorphic to the n-dimensional cube*;
3) *for every closed subset $F \subset X$ such that $f(F)$ is a nondegenerate continuum, $F = f^{-1}(f(F))$*;
4) *f is continuous, irreducible and fully closed*;
5) *$Cl((f^{n_k})^{-1}(E_k)) = (f^{n_k})^{-1}(Cl(E_k))$ for every k.*

Proof. Put

$$G^{(k)}_{j_1,\dots,j_m} = \{y = (y_1, \dots, y_{n_k}) \in Y^{n_k}\colon y_i = a \Leftrightarrow i \in \{j_1, \dots, j_m\}\},$$

where $j_1, \dots, j_m$ are distinct positive integers, $j_l \leqslant n_k$ and $1 \leqslant m \leqslant n_k$.

In each set $G^{(k)}_{j_1,\dots,j_m} \cap Cl(E_k)$ we take some countable dense subset $D^{(k)}_{j_1,\dots,j_m}$ and put

$$D = \bigotimes \{D^{(k)}_{j_1,\dots,j_m}\colon k \in N,\ 1 \leqslant m \leqslant n_k,\ j_i \leqslant n_k,\ j_i \neq j_l \text{ for } i \neq l\}.$$

Enumerate the points of D by positive integers: $D = \{d^i\}$. Each point d^i in D belongs to exactly one set $D^{(k)}_{j_1,\dots,j_m}$ whose indexes depend on i: $k = k(i)$, $m = m(i)$, $j_l = j^i_l$, $l = 1, \dots, m(i)$. By construction

$$D^{(k)}_{j_1,\dots,j_m} \subset G^{(k)}_{j_1,\dots,j_m} \cap Cl(E_k).$$

Therefore, for every i there is a sequence $C_i \subset E_{k(i)}$ converging to d^i. Put

$$H = \bigcup_i \bigcup_{l=1}^{m(i)} p_{j^i_l} C_i.$$

In [6] it is shown that we can choose subsequences $C'_i \subset C_i$ so that the set H will be a sequence converging to a and $p_{j^i_l} C'_i \cap p_{j^{i'}_{l'}} C'_{i'} = \emptyset$ for $i \neq i'$ and l, l' such that $1 \leqslant l \leqslant m(i), 1 \leqslant l' \leqslant m(i')$. To simplify notation we will denote these subsequences also by C_i.

Let us endow the set $f^{-1}(a)$ with topology of n-dimensional cube I^n. Let $G = \{r_k\colon k \in N\}$ be a countable dense subset of I^n. For any $m \in N$ we enumerate the points of N^m by positive integers. Let $k_1, \ldots, k_m$ be the coordinates of a point of N^m with the number k. Take a countable open base $\{O_k\colon k \in N\}$ at the point a in Y such that $Cl(O_{k+1}) \subset O_k$ and $Fr(O_k) \cap H = \emptyset$ for every $k \in N$. Fix an enumeration of points of each sequence $C_i = \{x^i_k\colon k \in N\}$. In this way we fix also an enumeration in each projections of C_i into Y. Define a mapping

$$h : H \cup \bigcup_{k \in N} Fr(O_k) \to I^n$$

as follows: if x^i_k is a point of the sequence C_i then

$$h\big(p_{j^i_l}(x^i_k)\big) = r_{k_l},$$

where k_l is the l-coordinate of a point of the set $N^{m(i)}$ with number k and $l = 1, \ldots, m(i)$; if $y \in Fr(O_k)$ then $h(y) = r_k$. It is easy to verify that the mapping h is well defined and continuous on $H \cup \bigcup_{k \in N} Fr(O_k)$. Since $H \cup \bigcup_{k \in N} Fr(O_k)$ is closed in $Y \setminus \{a\}$ the mapping h extends to a continuous mapping $h_a : Y \setminus \{a\} \to I^n$.

Let us now define a topology on X. If $x \in X$ and $f(x) \neq a$ then we take an open base at x as follows: $\{f^{-1}(U)\colon U$ is open in $Y,\ f(x) \in U\}$. If $f(x) = a$ then an open base at x consists of sets of the form

$$V \cup f^{-1}\big(U \cap h_a^{-1}(V)\big)$$

where U is a neighborhood of a in Y and V is a neighborhood of x in $I^n = f^{-1}(a)$. In fact the space X is the resolution of Y with one nontrivial fibre $f^{-1}(a) = I^n_a$ by the mapping h_a (see [2]). Hence, X is compact and f is continuous and fully closed; X is metrizable since $w(X) = \omega_0$. Conditions 1), 3), 4) of the lemma follow from the definition of the mapping h_a (see also [1,2]).

Let us verify condition 5). Pick $y \in Cl(E_k)$. If no coordinate of y is equal to a then the mapping f^{n_k} is one-to-one at y and consequently $(f^{n_k})^{-1} y \subset Cl((f^{n_k})^{-1} E_k)$.

Suppose now that $y = (y_1, \ldots, y_{n_k}) \in Cl(E_k)$ has some coordinates equal to a. Then y belongs to some $G^{(k)}_{j_1,\ldots,j_m}$. Let $x \in (f^{n_k})^{-1}(y)$ and let Ox be a neighborhood of x. Let us show that $(f^{n_k})^\sharp Ox \cap E_k \neq \emptyset$. Then $Ox \cap (f^{n_k})^{-1} E_k \neq \emptyset$ and, consequently, $x \in Cl((f^{n_k})^{-1} E_k)$.

Take a neighborhood O of x such that $O \subset Ox$ and

$$O = \prod_{j=1}^{n_k} Ox_j$$

where Ox_j are base neighborhoods of coordinates of $x = (x_1, \ldots, x_{n_k})$. Note that $(f^{n_k})^\sharp O = \prod f^\sharp Ox_j$ and $U_j = f^\sharp Ox_j$ are neighborhoods of y_j for $j \neq j_1, \ldots, j_m$. For $j = j_l$ $(l = 1, \ldots, m)$ $x_{j_l} \in I^n = f^{-1}(a)$ and

$$Ox_{j_l} = V_{j_l} \cup f^{-1}\big(U_{j_l} \cap h_a^{-1}(V_{j_l})\big)$$

where V_{j_l} is a neighborhood of x_{j_l} in I^n and U_{j_l} is a neighborhood of a in Y. The set

$$\prod_{j=1}^{n_k} U_j$$

is a neighborhood of y in Y^{n_k}, consequently, there is a point

$$d^i \in D^{(k)}_{j_1,\ldots,j_m} \subset D$$

such that $d^i \in U$. The sequence $C_i = \{x^i_{k'}\colon k' \in N\} \subset E_k$ constructed above converges to d^i. Hence beginning with some position m_0 all members of this sequence belong to U.

Let

$$(r_{q_1}, \ldots, r_{q_m}) \in \prod_{l=1}^{m} V_{j_l} \subset \big(I^n\big)^m$$

where $r_{q_l} \in G \subset I^n$, $l = 1, \ldots, m$ and let the point $(q_1, \ldots, q_m) \in N^m$ have a number $k' > m_0$ in the chosen numeration of N^m. Then $q_l = k'_l$, $l = 1, \ldots, m$. By the definition of the mapping h_a

$$h_a\big(p_{j_l}(x^i_{k'})\big) = r_{k'_l}.$$

Consequently, $p_{j_l}(x^i_{k'}) \in f^\sharp(Ox_{j_l})$, $l = 1, \ldots, m$ and $x^i_{k'} \in (f^{n_k})^\sharp O$. The proof of the lemma is complete. $\square$

Let $X_0 = I^n$ and let L be a Lusin set in X_0. The latter means that L is uncountable and every uncountable subset L is somewhere dense in X_0. Each metrizable compact space without isolated points has a Lusin set under the assumption of *CH*. Enumerate the points of L by countable ordinals: $L = \{x_\beta\colon\ \beta < \omega_1\}$. Put $L_\alpha = \{x_\beta\colon\ \beta < \alpha\}$.

Let $p : X_0 \times I^n \to X_0$ be a projection of sets without topologies. Given $\alpha \leqslant \omega_1$, define the equivalence relation R_α on $X_0 \times I^n$ as follows:

$$\text{if } x \neq y \ \big(x, y \in X_0 \times I^n\big), \quad \text{then } xR_\alpha y \ \Leftrightarrow\ p(x) = p(y) \in X_0 \setminus L_\alpha.$$

Put $X_\alpha = X_0 \times I^n / R_\alpha$. For $\alpha > \beta$ quotient sets X_α and X_β are related with the natural projection $p^\alpha_\beta : X_\alpha \to X_\beta$. The system of sets and mappings $S = \{X_\alpha, p^\alpha_\beta\colon\ \alpha, \beta < \omega_1\}$ is an inverse system, whose limit $\lim S = X$ can be identified with X_{ω_1} and whose limit projections $p_\alpha : X \to X_\alpha$ can be identified with mappings $p^{\omega_1}_\alpha$.

Let $\alpha < \omega_1$ and $m \in N$. A countable set $E \subset X^m_\alpha$ is called αm-admissible if E is in general position in X^m_α and $|(p^m_\alpha)^{-1}x| = 1$ for every point $x \in E$. Denote by $A_{\alpha m}$ the set of all αm-admissible sets and put

$$A = \bigcup \{A_{\alpha m}\colon\ \alpha < \omega_1,\ m \in N\}.$$

Since $|A| = c$, assuming *CH*, we can enumerate A with countable ordinals $A = \{E_\beta\colon\ \beta < \omega_1\}$. Here we require that if E_β is an αm-admissible set then $\beta \geqslant \alpha$.

We now use recursion to topologize the elements of the inverse system S. The set X_0 is endowed with the topology of I^n. Suppose that for every $\beta < \alpha$ the set X_β is already endowed with a topology so that

(1_α) $S_\alpha = \{X_\beta, p^\beta_{\beta'}\colon\ \beta, \beta' < \alpha\}$ is a continuous inverse system with irreducible projections and connected compact spaces;

(2_α) if $\beta + 1 < \alpha$ and E_δ ($\delta \leqslant \beta$) is a γm-admissible set then

$$Cl\big(\big(\big(p^{\beta+1}_\gamma\big)^m\big)^{-1} E_\delta\big) = \big(\big(p^{\beta+1}_\beta\big)^m\big)^{-1} Cl\big(\big(\big(p^\beta_\gamma\big)^m\big)^{-1} E_\delta\big);$$

(3_α) if $\beta + 1 < \alpha$ then $(p^{\beta+1}_0)^{-1} x_\beta = I^n$ and $p^{\beta+1}_\beta$ is fully closed;

(4_α) if $\beta + 1 < \alpha$ and F is a closed subset of $X_{\beta+1}$ such that $p^{\beta+1}_\beta(F)$ is connected and $|p^{\beta+1}_\beta(F)| > 1$ then $F = (p^{\beta+1}_\beta)^{-1}(p^{\beta+1}_\beta(F))$.

For a limit ordinal α endows X_α with the topology of limit of the inverse system S_α: $X_\alpha = \lim S_\alpha$. It is obvious that the conditions ($1_{\alpha+1}$)–($4_{\alpha+1}$) will be fulfilled.

Now consider the case $\alpha = \xi + 1$. In this case define a topology on X_α using the lemma proved above, where as f we take the mapping $p^\alpha_\xi : X_\alpha \to X_\xi$ and collect into the countable family of $\{E_k\}$ all sets of the form $((p^\xi_\gamma)^m)^{-1} E_\delta$ where $\delta \leqslant \xi$ and E_δ is γm-admissible. The conditions 1)–5) of the lemma imply that ($1_{\alpha+1}$)–($4_{\alpha+1}$) are fulfilled.

In result of this topologization we obtain a continuous inverse system $S = \{X_\alpha, p^\alpha_\beta\colon\ \alpha, \beta < \omega_1\}$ of compact spaces that satisfies the conditions (1_{ω_1})–(4_{ω_1}). The limit of this inverse system is the required compact space X_n. The weight of X_n is uncountable since all projections of S are not homeomorphisms.

Let us verify condition (1) of the theorem. Inverse system S satisfies the conditions of Corollary 4.2 in [2]. Hence $\dim X_n \leqslant n$. On the other hand for every $x_\alpha \in L \subset X_0$ we have $p_0^{-1}(x_\alpha) = I^n \subset X_n$; therefore $\dim X_n \geqslant n$.

Let P be a partition in X_n, i.e. $X \setminus P = U_1 \cup U_2$ where U_1, U_2 are nonempty disjoint open sets in X_n. The mapping p_0 is monotone and irreducible, hence

$$X_0 = p_0(P) \cup p_0^\sharp U_1 \cup p_o^\sharp U_2,$$

where $p_0^\sharp U_i \neq \emptyset$. Hence $p_0(P)$ is a partition in $X_0 = I^n$; therefore $\dim p_0(P) \geqslant n-1$. There is a connected $(n-1)$-dimensional subset $\Phi \subset p_0(P)$. Put $F = p_0^{-1}(\Phi) \cap P$. Condition ($4_{\omega_1}$) implies that

$$p_0^{-1}(\Phi) = p_0^{-1}\big(p_0(F)\big) = F.$$

If $\Phi \cap L = \emptyset$ then F is homeomorphic to Φ and $\dim F = n-1$. If there exists $x_\alpha \in L \cap \Phi$ then $I^n = p_0^{-1}(x_\alpha) \subset F$ and $\dim F \geqslant n$. Hence $\dim P \geqslant n-1$.

Condition (2_{ω_1}) implies the following statement (see [6]):

(F) if E_δ is an αm-admissible then for every $\beta \geqslant \delta$

$$Cl\big(\big(p^m_\alpha\big)^{-1} E_\delta\big) = \big(p^m_\beta\big)^{-1} Cl\big(\big(\big(p^\beta_\alpha\big)^m\big)^{-1} E_\delta\big).$$

Let us verify condition (2) of the theorem. It is known that if the square of a compact space X is a hereditarily Suslin space then X is hereditarily separable (see [5]). Therefore, it suffices to verify that for every m in X^m_n there is no uncountable discrete subspace. We use induction to proof this statement.

$m = 1$. Suppose that there exists an uncountable discrete subset $D \subset X_n$. Since $|p_0^{-1}(x)| = 1$ for $x \notin L$ and $p_0^{-1}(x) = I^n$ for $x \in L$, without loss of generality we may assume that p_0 is one-to-one on D. Let G be a countable dense subset of $p_0(D)$.

Put $E' = p_0^{-1}G \cap D$. Take $\alpha < \omega_1$ such that $p_\alpha^{-1}(p_\alpha(E')) = E'$. Then $E = p_\alpha(E')$ is an $\alpha 1$-admissible; therefore $E = E_\beta$ for some $\beta \geqslant \alpha$. Because of (F) we have

$$Cl\big(p_\alpha^{-1}(E)\big) = p_\beta^{-1}\big(Cl\big(\big(p_\alpha^\beta\big)^{-1}(E)\big)\big). \tag{1}$$

The mapping p_0^β is one-to-one at each point $x \in p_\beta D$ for which $p_0^\beta(x) \notin L_\beta$. Therefore, if such a point x is not in $(p_\alpha^\beta)^{-1}(E)$ then x is a limit point of $(p_\alpha^\beta)^{-1}(E)$. Then by (1) the point $x' \in D$ "covering" x is a limit point of E'; this is a contradiction since D is discrete.

Suppose that our statement is proved for $k < m$ and let D' be an uncountable discrete subset of X_n^m. In this situation it is easy to construct an uncountable subset $D \subset D'$ that is in general position in X_n^m (see [6]). For each point $x = (x_1, \dots, x_m) \in X_0^m$ the fibre $(p_0^m)^{-1}(x)$ is homeomorphic to $(I^n)^k$ where $k = |\{i\colon\ x_i \in L\}|$. Therefore the intersection $(p_0^m)^{-1}(x) \cap D$ is countable for any $x \in X_0^m$. Hence without loss of generality we may assume that p_0 is one-to-one on D. Let G be a countable dense subset of $p_0^m(D)$. Put $E' = (p_o^m)^{-1}G \cap D$. Take $\alpha < \omega_1$ such that $(p_\alpha^m)^{-1}(p_\alpha^m(E')) = E'$. Then $E = p_\alpha^m(E')$ is an αm-admissible; therefore $E = E_\beta$ for some $\beta \geqslant \alpha$. Because of (F) we have

$$Cl\big(\big(p_\alpha^m\big)^{-1}(E)\big) = \big(p_\beta^m\big)^{-1}\big(Cl\big(\big(\big(p_\alpha^\beta\big)^m\big)^{-1}(E)\big)\big). \tag{2}$$

Put

$$M = \big\{x \in p_\beta^m(D)\colon\ \big|\big(\big(p_o^\beta\big)^m\big)^{-1}\big(p_0^\beta\big)^m(x)\big| > 1\big\}.$$

Let us prove that $|M| \leqslant \omega_0$. Assume the contrary. Observe that if $x \in M$ then there exists a coordinate of a point $(p_0^\beta)^m(x)$ that is in L_β. Since L_β is countable, there is a point $x_\gamma \in L_\beta$ such that all points of some uncountable subset $D'' \subset D$ have a coordinate with fixed number in the fibre $p_0^{-1}(x_\gamma) = I^n$. Then $D'' \subset I^n \times X_n^{m-1}$; this is a contradiction with inductive assumption.

Since G is dense in $p_o^m(D)$ and $(p_0^\alpha)^m(E) = G$ each point $x \in p_\beta^m(D) \setminus (((p_\alpha^\beta)^m)^{-1}(E) \cup M)$ is a limit point of $((p_\alpha^\beta)^m)^{-1}(E)$. Then by (2) the point $x' \in D$ "covering" x is a limit point of E'.

The verification of conditions (3), (4), (5) coincides with the verification of conditions (2), (3), (4) of Theorem 1 in [6].

Let us verify condition (6). Let $\mathcal{F}$ be a seminormal functor that preserves weights and one-to-one points and $sp(\mathcal{F}) = \{1, k\}$. Then $\mathcal{F}(X_n) = \lim \mathcal{F}(S)$ where

$$\mathcal{F}(S) = \big\{\mathcal{F}(X_\alpha), \mathcal{F}(p_\beta^\alpha)\colon\ \alpha, \beta < \omega_1\big\}.$$

A proof of hereditary normality of $\mathcal{F}(X_n)$ is contained in [6]. It is only necessary to verify that for every $x_\gamma \in L_\beta$ the fibre $(\mathcal{F}(p_0))^{-1}(x_\gamma)$ is perfectly normal. The space $(\mathcal{F}(p_0))^{-1}(x_\gamma) \setminus X_n$ is perfectly normal since it is a subset of the perfectly normal space $\mathcal{F}_{kk}(X_n) = \mathcal{F}(X_n) \setminus X_n$. We have

$$\big(\mathcal{F}(p_0)\big)^{-1}(x_\gamma) \cap X_n = p_0^{-1}(x_\gamma) = I^n$$

and

$$p_0^{-1}(x_\gamma) = p_{\gamma+1}^{-1}\big(\big(p_0^{\gamma+1}\big)^{-1}x_\gamma\big) = p_{\gamma+1}^{-1}\big(I^n\big).$$

The mapping $p_{\gamma+1}$ is one-to-one at each point $x \in I^n = (p_0^{\gamma+1})^{-1}(x_\gamma) \subset X_{\gamma+1}$. Hence

$$p_0^{-1}(x_\gamma) = I^n = \big(\mathcal{F}(p_{\gamma+1})\big)^{-1}\big(\big(p_0^{\gamma+1}\big)^{-1}(x_\gamma)\big).$$

Therefore the set $(\mathcal{F}(p_0))^{-1}(x_\gamma) \cap X_n = I^n$ is a G_δ-set in $\mathcal{F}(X_n)$. It follows that $(\mathcal{F}(p_0))^{-1}(x_\gamma)$ is perfectly normal. This completes the proof of the theorem. □

Remark. If $\mathcal{F}$ is a seminormal functor that preserves one-to-one points and $sp(\mathcal{F}) = \{1, k\}$ then $k \leqslant 3$.

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